Math 246C Lecture 23 Notes

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1 Hömander's Theorem for Solving the $\overline{\partial}$-Equation in One Variable

1.1 Completion of the proof of Hömander's theorem

We want to solve $\overline{\partial} u = f$ on $\Omega \subseteq \mathbb{C}$. Last time, we were proving the following observation:

Proposition 1.1. *Let $f \in L^2_{\mathrm{loc}}(\Omega)$, and let $C > 0$ be constant. Then there exists a $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $\overline{\partial} u = f$ and $\int |u|^2 e^{-\varphi}\, L(dz) \leq C$ if and only if*

$$\left| \int f \overline{\beta} e^{-\varphi}\, L(dz) \right| \leq C \int |\overline{\partial}^*_\varphi \beta|^2 e^{-\varphi}\, L(dz) \qquad \forall \beta \in C_0^\infty(\Omega).$$

Proof. ($\Longleftarrow$): Consider the linear map $F : \overline{\partial}^*_\varphi C_0^\infty(\Omega) \to \mathbb{C}$ given by$F(\overline{\partial}^*_\varphi \beta) = \int f \overline{\beta} e^{-\varphi}$. Then

$$|F(\overline{\partial}^*_\varphi \beta)| \leq C^{1/2} \|\overline{\partial}^*_\varphi \beta\|_{L^2},$$

By the Hahn-Banach theorem, F extends to a linear continuous map on L^2_φ with the norm $\leq C^{1.2}$. Thus, there exists a $u \in L^2_\varphi$ with $\|u\|_{L^2_\varphi} \leq C^{1/2}$ such that $F(g) = \langle g, h \rangle_{L^2_\varphi}$ for all $g \in L^2 \varphi$. In particular, if $g = \overline{\partial}^*_\varphi \beta$,

$$\overline{\int f \overline{\beta} e^{-\varphi}} = \langle \overline{\partial}^*_\varphi \beta, u \rangle_{L^2_\varphi} \qquad \forall \beta \in C_0^\infty.$$

We get

$$\int f\, \overline{\beta} e^{-\varphi} = -\int u \partial_{\overline{z}} (e^{-\varphi \overline{\beta}}).$$

for all β. So we get that $\overline{\partial} u = f$ weakly. $\square$

We can now complete the proof of Hörmander's theorem.

Theorem 1.1 (Hörmander[1]). *Let $\Omega \subseteq \mathbb{C}$ be open, and let $\varphi \in C^\infty(\Omega)$ be strictly subharmonic: $\Delta\varphi > 0$ in Ω. Then, for any $f \in L^2_{\mathrm{loc}}(\Omega)$ such that*

$$\int \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi}\, L(dz) < \infty,$$

there exists a weak solution $u \in L^2_{\mathrm{loc}}(\Omega)$ to $\frac{\partial u}{\partial \overline{z}} = f$ such that

$$\int_\Omega |u|^2 e^{-\varphi}\, L(dz) \le \int_\Omega \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi}\, L(dz).$$

Proof. We need to show that

$$\left| \int f\overline{\beta} e^{-\varphi} \right|^2 \le C \|\overline{\partial}^*_\varphi \beta\|^2_{L^2_\varphi} \qquad \forall \beta \in C_0^\infty.$$

We need a lower bound for $\|\overline{\partial}^*_\varphi \beta\|^2_{L^2_\varphi}$:

In general, let H be a Hilbert space, and let $T \in \mathcal{L}(H, H)$. Then

$$\begin{aligned} \|T^*x\|^2 \ge \|T^*x\|^2 - \|Tx\|^2 &= \langle T^*x, T^*x\rangle - \langle Tx, Tx\rangle \\ &= \langle TT^*x, x\rangle - \langle T^*Tx, x\rangle \\ &= \langle [T, T^*]x, x\rangle, \end{aligned}$$

where $[T, T^*] = TT^* - T^*T$ is the commutator of T, T^*. In our case, $H = L^2_\varphi$, $T = \overline{\partial}$, and $T^* = \overline{\partial}^*_\varphi = -\partial_z + \partial_z\varphi$. So The commutator is

$$[\overline{\partial}, \overline{\partial}^*_\varphi] = [\overline{\partial}, -\partial + \partial\varphi] = \cancelto{0}{[\overline{\partial}, \partial]} + [\overline{\partial}, \partial\varphi].$$

Compute for $\beta \in C_0^\infty$:

$$[\overline{\partial}, \partial\varphi]\beta = \overline{\partial}(\partial\varphi\beta) - \partial\varphi\overline{\partial}\beta = \underbrace{(\overline{\partial}\partial\varphi)}_{\Delta\varphi/4>0}\, \beta.$$

We get that

$$\|\overline{\partial}^*_\varphi \beta\|^2_{L^2_\varphi} \ge \frac{1}{4} \int \Delta\varphi |\beta|^2 e^{-\varphi} \qquad \forall \beta \in C_0^\infty(\Omega).$$

It follows by Cuachy-Schwarz in L^2_φ that

$$\left| \int f\overline{\beta} e^{-\varphi} \right| \le \left(\int \frac{|f|^2}{\Delta\varphi} e^{-\varphi} \right) \underbrace{\left(\int \Delta\varphi |\beta|^2 e^{-\varphi} \right)}_{\le 4\|\overline{\partial}^*_\varphi\beta\|^2_{L^2_\varphi}}.$$

[1] This result, unlike the other results we have been proving, is fairly recent. It was proven in 1965.

Finally, we get that there exists some $u \in L^2_\varphi$ such that $\overline{\partial} u = f$ and

$$\|u\|^2_{L^2_\varphi} \leq 4 \int \frac{|f|^2}{\Delta\varphi} e^{-\varphi}. \qquad \square$$

Remark 1.1. $\overline{\partial}^*_\varphi C_0^\infty(\Omega) \subseteq L^2_\varphi$: we obtain $u \in \overline{\overline{\partial}^*_\varphi C_0^\infty(\Omega)}$ such that if $h \in \ker(\overline{\partial}) \cap L^2_\varphi$ (i.e. h is holomorphic), then

$$0 = \langle \overline{\partial} h, \beta \rangle = \langle h, \overline{\partial}^*_\varphi \beta \rangle_{L^2_\varphi} \qquad \forall \beta \in C_0^\infty.$$

So $u \perp \ker(\overline{\partial}) \cap L^2_\varphi$. Thus, we have found a solution of $\overline{\partial} u = f$ of minimal norm in L^2_φ.

1.2 Weakening the assumptions of Hörmander's theorem

Assume that $\varphi \in C^\infty(\Omega)$ is just subharmonic: $\Delta\varphi \geq 0$. Apply Hörmander's theorem to

$$\psi(z) = \varphi(z) + a \log(1 + |z|^2), \qquad a > 0.$$

We can estimate (setting $r = |z|$):

$$\Delta\psi(z) \geq a \underbrace{\Delta \log(1 + |z|^2)}_{=(\partial_r^2 + \frac{1}{r}\partial_r)(\log(1+r^2)))} = \frac{4}{(1+r^2)^2}.$$

We get that $\overline{\partial} u = f$ has a solution $u \in L^2_{\text{loc}}$ such that

$$a \int_\Omega |u|^2 e^{-\varphi} (1 + |z|^2)^{-a} \leq \int |f|^2 e^{-\varphi} (1 + |z|^2)^{2-a}$$

for all subharmonic $\varphi \in C^\infty$.

It turns out that the same estimate is valid for any subharmonic function, not just ones in C^∞.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\varphi \in \mathrm{SH}(\Omega)$ with $\varphi \not\equiv -\infty$. Let $a > 0$, and assume that $f \in L^2_{\text{loc}}$ is such that*

$$\int |f|^2 e^{-\varphi} (1 + |z|^2)^{2-a} < \infty.$$

Then there exists a u solving $\overline{\partial} u = f$ such that

$$a \int_\Omega |u|^2 e^{-\varphi} (1 + |z|^2)^{-a} \leq \int |f|^2 e^{-\varphi} (1 + |z|^2)^{2-a}.$$

We will prove this next time.

Remark 1.2. Let $f \in L^2_{\text{loc}}(\Omega)$. Then there is a $u \in L^2_{\text{loc}}(\Omega)$ solving $\overline{\partial} u = f$: there exists a $\varphi \in C(\Omega) \cap \mathrm{SH}(\Omega)$ such that $f \in L^2(\Omega, e^{\varphi})$ (that is, $\int |f|^2 e^{-\varphi} < \infty$: for $\Omega \neq \mathbb{C}$, take

$$\varphi_0(z) = -\log(\operatorname{dist}(z, \Omega^c)),$$

which is subharmonic in Ω with the property that $\varphi_0(z) \to \infty$ as $z \to \partial\Omega$. Composing φ_0 with a suitable convex increasing function, we get φ such that the bound holds.